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On Hermite–Fejér Interpolation at Jacobi Zeros

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DEDICATED TO MY TEACHER, PROFESSOR DR. K. ZELLER, UNIVERSITY OF TÜBINGEN, ON THE OCCASION OF HIS SIXTIETH BIRTHDAY (DECEMBER 28, 1984)

For the Hermite–Fejér interpolation at the zeros of the Jacobi polynomials $P_m^{(\alpha,\beta)}$ it is shown, with the aid of the Bohman–Korovkin theorem, that the sequence of interpolation polynomials converges for every continuous f, pointwise, for $|x|<1$ and $\alpha, \beta > -1$ and that the convergence is uniform in every closed subinterval $[-1+\delta, 1-\delta]$, $\delta>0$. Moreover, there is uniform convergence for $|x| \leqslant 1$ if $\max(\alpha, \beta)<0$. To prove this result of Szegö, we show that the Hermite–Fejér functionals are asymptotically positive and that the application of a functional of that type to the test function $g_x: t \to (x-t)^2$ yields a constant multiple of $\{P_m^{(\alpha,\beta)}\}^2$. Our methods may also be used to prove an analogous convergence result for the generalized Hermite–Fejér interpolation with Jacobi nodes of multiplicity 4.

1. INTRODUCTION

The aim of this article is to show that the convergence results of Szegö [8, 9] for the Hermite–Fejér interpolation may be proved via the Bohman–Korovkin theorem on the convergence of sequences of positive linear functionals, taking advantage of special properties of Jacobi polynomials (bounds for the zeros and the sup-norm, Gaussian quadrature). Fejér [1] and Szegö [8, 9] proved their convergence theorems for Hermite–Fejér interpolation directly without the elegant mechanism of the Bohman–Korovkin theory which was not yet known at that time; nevertheless positivity arguments played a central role in their proofs. Later the Bohman–Korovkin theorem was used as a powerful method for an elegant proof of the convergence results of Fejér and Szegö in case of $P_m^{(\alpha,\beta)}$-nodes with $\max(\alpha, \beta)<0$ and for some generalizations of Lobatto type where the Hermite–Fejér operators are positive (cf. de Vore

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[10], Knoop [4]). But the possibility to prove Szegö's convergence theorem for arbitrary $\alpha, \beta > -1$ and $|x| < 1$ with the aid of the Bohman–Korovkin theorem seems to be remained unnoticed until now.

In this article we show that the Hermite–Fejér functional at a fixed position x with $|x| < 1$ and for arbitrary Jacobi nodes may be splitted into a positive functional and a perturbation term which has arbitrary small norm if the number of nodes increases. In this sense the Hermite–Fejér functionals are asymptotically positive and this property in connection with the convergence for the test function $g_x: t \to (x-t)^2$ guarantees the convergence of the Hermite–Fejér interpolation for all continuous functions. In this context we point to the fact that an application of the Hermite–Fejér operator to g_x followed by an evaluation at the point x yields a polynomial which is a constant multiple of $[P_m^{(\alpha,\beta)}]^2$. In this way the convergence for the test function g_x is easily proved. Altogether we get the convergence theorem of Szegö, namely that for every continuous f the sequence of Hermite–Fejér interpolation polynomials to the zeros of the Jacobi polynomial $P_m^{(\alpha,\beta)}$ converges to f pointwise for all x with $|x| < 1$ and uniform in every closed subinterval provided $\alpha, \beta > -1$; the convergence is uniform in $|x| \leqslant 1$ if $\alpha, \beta > -1$ and $\max(\alpha, \beta) < 0$. If we generalize the Hermite–Fejér problem allowing nodes of multiplicity 4, we get by similar arguments analogous results; the generalized Hermite–Fejér interpolation converges pointwise for $|x| < 1$ and uniform in closed subintervals for arbitrary $\alpha, \beta \geqslant -\frac{3}{4}$ and the convergence is uniform in $|x| \leqslant 1$ if max $(\alpha, \beta) < -\frac{1}{4}$. We point to the fact that with respect to the parameters α, β our convergence results cover the known convergence areas.

2. Convergence for the Test Function

We start with the definition of the Hermite operator

$$H_m: C^1[-1, 1] \to \pi_{2m-1},$$

where $x_\mu = x_\mu^{(m)}$, $\mu = 1,\ldots, m$, is the set of nodes with $-1 \leqslant x_1 < \cdots < x_m \leqslant 1$ and

$$H_m f = \sum_{\mu=1}^{m} \{f(x_\mu)\, l_{\mu 0} + f'(x_\mu)\, l_{\mu 1}\},$$

$$l_{\mu 0}(x) = \left\{1 - \frac{\omega_m''(x_\mu)}{\omega_m'(x_\mu)}(x - x_\mu)\right\} \{l_\mu(x)\}^2,$$

$$l_{\mu 1}(x) = (x - x_\mu)\{l_\mu(x)\}^2,$$

$$\omega_m(x) = \prod_{\rho=1}^{m} (x - x_\rho),$$

$$l_\mu(x) = \frac{\omega_m(x)}{\omega_m'(x_\mu)(x - x_\mu)}.$$

The associated Hermite–Fejér operator

$$F_m: C[-1, 1] \to \pi_{2m-1}$$

then has the representation

$$F_m f = \sum_{\mu=1}^{m} f(x_\mu)\, l_{\mu 0}.$$

To determine $F_m g_x$, $g_x: t \to (x-t)^2$, we note that

$$H_m g_x = g_x, \qquad m \geqslant 2.$$

Then it follows

$$g_x = F_m g_x - 2 \sum_{\mu=1}^{m} (x - x_\mu)\, l_{\mu 1}.$$

As $g_x(x) = 0$, we get

$$\begin{aligned}(F_m g_x)(x) &= 2 \sum_{\mu=1}^{m} (x - x_\mu)\, l_{\mu 1}(x) \\ &= 2 \sum_{\mu=1}^{m} (x - x_\mu)^2 \{l_\mu(x)\}^2 \\ &= \{\omega_m(x)\}^2 \sum_{\mu=1}^{m} \frac{2}{[\omega_m'(x_\mu)]^2}.\end{aligned}$$

Now we consider the special case of Jacobi nodes and use the notation $F_m^{(\alpha,\beta)}$ instead of F_m. Let

$$\omega_m = \binom{m+\alpha}{m}^{-1} P_m^{(\alpha,\beta)}$$

with $\alpha, \beta > -1$. Then in the relation

$$(F_m^{(\alpha,\beta)} g_x)(x) = \{P_m^{(\alpha,\beta)}(x)\}^2 \sum_{\mu=1}^{m} \frac{2}{[P_m^{(\alpha,\beta)\prime}(x_\mu)]^2}$$

the sum is related to the weights

$$\lambda_\mu = 2^{\alpha+\beta+1} \frac{\Gamma(m+\alpha+1)\,\Gamma(m+\beta+1)}{m!\,\Gamma(m+\alpha+\beta+1)} (1-x_\mu^2)^{-1} \{P_m^{(\alpha,\beta)\prime}(x_\mu)\}^{-2}$$

of Gauss–Jacobi quadrature (cf. Szegö [9, p. 352]). It follows

$$(F_m^{(\alpha,\beta)} g_x)(x) = \{P_m^{(\alpha,\beta)}(x)\}^2\, c_m^{(\alpha,\beta)} \sum_{\mu=1}^{m} \lambda_\mu (1-x_\mu^2)$$

with

$$c_m^{(\alpha,\beta)} := 2^{-\alpha-\beta} \frac{m!\,\Gamma(m+\alpha+\beta+1)}{\Gamma(m+\alpha+1)\,\Gamma(m+\beta+1)}.$$

As

$$\sum_{\mu=1}^{m} \lambda_\mu (1-x_\mu^2) = \int_{-1}^{1} (1-t^2)(1-t)^\alpha (1+t)^\beta\, dt$$
$$= 2^{\alpha+\beta+3} \frac{\Gamma(\alpha+2)\,\Gamma(\beta+2)}{\Gamma(\alpha+\beta+4)}$$

we get

LEMMA 1. *If the Hermite–Fejér operator $F_m^{(\alpha,\beta)}$ is applied to the test function $g_x = (x - \cdot)^2$ it follows for arbitrary $\alpha, \beta > -1$*

$$(F_m^{(\alpha,\beta)} g_x)(x) = \sigma_m^{(\alpha,\beta)} \{P_m^{(\alpha,\beta)}(x)\}^2,$$

where

$$\sigma_m^{(\alpha,\beta)} := 8 \frac{\Gamma(\alpha+2)\,\Gamma(\beta+2)}{\Gamma(\alpha+\beta+4)} \frac{m!\,\Gamma(m+\alpha+\beta+1)}{\Gamma(m+\alpha+1)\,\Gamma(m+\beta+1)}$$
$$= O(1) \quad \text{as} \quad m \to \infty.$$

An application of a result of Szegö [9, Theorem 7.32.2, p. 169] now yields

THEOREM 1. *One has as $m \to \infty$*

(1) $(F_m^{(\alpha,\beta)} g_x(x) = o(1)$, *pointwise for all x with $|x| < 1$, if $\alpha, \beta > -1$,*

(2) $(F_m^{(\alpha,\beta)} g_x)(x) = o(1)$, *uniform for all x with $|x| \leqslant 1-\delta, \delta > 0$, if $\alpha, \beta > -1$, and*

(3) $(F_m^{(\alpha,\beta)} g_x)(x) = o(1)$, *uniform for all x with $|x| \leqslant 1$, if $\alpha, \beta > -1$ and* $\max(\alpha, \beta) < 0$.

Remark 1. Since

$$P_m^{(-1/2,-1/2)} = \binom{m-1/2}{m} T_m,$$

we get from Lemma 1 the well-known relation (cf. de Vore [8, p. 43])

$$(F_m^{(-1/2,-1/2)} g_x)(x) = \frac{1}{m} \{T_m(x)\}^2.$$

3. Asymptotic Positivity of the Hermite–Fejér Functionals

Now we consider the Hermite–Fejér functional $F_{mx}^{(\alpha,\beta)}$ defined by

$$F_{mx}^{(\alpha,\beta)} f := (F_m^{(\alpha,\beta)} f)(x).$$

The starting point for our positivity considerations is the representation (cf. Szegö [9, (14.17) and (14.5.2)]

$$F_{mx}^{(\alpha,\beta)} f = \sum_{\mu=1}^{m} f(x_\mu)\, u_\mu(x) \{l_\mu(x)\}^2$$

with

$$u_\mu(x) = \frac{1 - x[\alpha - \beta + (\alpha+\beta+2)\, x_\mu] + (\alpha-\beta)\, x_\mu + (\alpha+\beta+1)\, x_\mu^2}{1 - x_\mu^2}.$$

For the operator norm of $F_{mx}^{(\alpha,\beta)}$, we get

$$\|F_{mx}^{(\alpha,\beta)}\| = \sum_{\mu=1}^{m} |u_\mu(x)|\ \{l_\mu(x)\}^2,$$

whence follows

$$\begin{aligned} \|F_{mx}^{(\alpha,\beta)}\| &= \sum_{\substack{\mu=1 \\ u_\mu(x) \geqslant 0}}^{m} u_\mu(x)\{l_\mu(x)\}^2 - \sum_{\substack{\mu=1 \\ u_\mu(x) < 0}}^{m} u_\mu(x)\{l_\mu(x)\}^2 \\ &= \sum_{\mu=1}^{m} u_\mu(x)\{l_\mu(x)\}^2 - 2 \sum_{\substack{\mu=1 \\ u_\mu(x) < 0}}^{m} u_\mu(x)\{l_\mu(x)\}^2 \\ &= 1 - 2 \sum_{\substack{\mu=1 \\ u_\mu(x) < 0}}^{m} u_\mu(x)\{l_\mu(x)\}^2. \end{aligned}$$

It is seen that $F_{mx}^{(\alpha,\beta)}$ is a positive functional in the usual sense if the sum is empty, and we say that $F_{mx}^{(\alpha,\beta)}$ is an *asymptotically positive functional* if

$$N_{mx}^{(\alpha,\beta)} := -2 \sum_{\substack{\mu=1 \\ u_\mu(x)<0}}^{m} u_\mu(x)\{l_\mu(x)\}^2 \to 0 \qquad \text{as } m \to \infty.$$

Since $u_\mu(x_\mu)=1$, one has $u_\mu(x)>0$ if $|x-x_\mu|$ is small. Then for fixed x with $|x| \leqslant 1$ there exist natural numbers $\mu=\mu(x)$ such that $u_\mu(x)>0$. These heuristics allow the conjecture that in an ε-neighbourhood of x no nodes with $u_\mu(x)<0$ lie where $\varepsilon>0$ and independent of m. To verify this conjecture, we consider the quadratic polynomial corresponding to the numerator of u_μ

$$\begin{aligned} p_x(t) &= 1 - x[\alpha-\beta+(\alpha+\beta+2)\,t] + (\alpha-\beta)\,t + (\alpha+\beta+1)\,t^2 \\ &= 1-x^2-[(\alpha+\beta+1)\,t+(\alpha-\beta-x)](x-t). \end{aligned}$$

For $|x|<1$, we have

$$\begin{aligned} p_x(-1) &= 2(\beta+1)(1+x)>0, \\ p_x(x) &= 1-x^2>0, \\ p_x(1) &= 2(\alpha+1)(1-x)>0. \end{aligned}$$

Thus, we have shown that from $u_\mu(x)<0$ the three estimates

$$|x-x_\mu| \geqslant \varepsilon, \qquad |\pm 1 - x_\mu| \geqslant \varepsilon$$

follow where $\varepsilon=\varepsilon(x,\alpha,\beta)>0$ and independent of m. For $|x|=1$, we consider only the case $\max(\alpha,\beta) \leqslant 0$, where $F_{mx}^{(\alpha,\beta)}$ is a positive functional so that $N_{mx}^{(\alpha,\beta)}=0$. Then it follows for $|x|<1$,

$$\|N_{mx}^{(\alpha,\beta)}\| = -2 \sum_{\substack{\mu=1 \\ u_\mu(x)<0}}^{m} u_\mu(x)\{l_\mu(x)\}^2 \leqslant$$

$$\sum_{\substack{\mu=1 \\ |x-x_\mu|\geqslant\varepsilon}}^{m} \left| \frac{[1-x[\alpha-\beta+(\alpha+\beta+2)\,x_\mu]+(\alpha-\beta)x_\mu+(\alpha+\beta+1)x_\mu^2][P_m^{(\alpha,\beta)}(x)]^2}{1/2(1-x_\mu^2)(x-x_\mu)^2\,[P_m^{(\alpha,\beta)\prime}(x_\mu)]^2} \right|$$

$$\leqslant \frac{2}{\varepsilon^2}\,[P_m^{(\alpha,\beta)}(x)]^2\,\rho \sum_{\substack{\mu=1 \\ |x-x_\mu|\geqslant\varepsilon}}^{m} \frac{1}{(1-x_\mu^2)[P_m^{(\alpha,\beta)\prime}(x_\mu)]^2},$$

where ρ is an upper bound of

$$|1-x[\alpha-\beta+(\alpha+\beta+2)\,x_\mu]+(\alpha-\beta)\,x_\mu+(\alpha+\beta+1)\,x_\mu^2|,$$

e.g.,

$$\rho = 1 + 2\,|\alpha - \beta| + (\alpha + \beta + 2) + |\alpha + \beta + 1|.$$

Moreover, if we compare to the Gauss–Jacobi quadrature formula, we get (cf. Szegö [9, (15.3.1)])

$$\begin{aligned}
&\sum_{\substack{\mu=1\\ |x-x_\mu|\geqslant\varepsilon}}^{m} \frac{1}{(1-x_\mu^2)[P_m^{(\alpha,\beta)\prime}(x_\mu)]^2}\\
&\quad\leqslant \sum_{\mu=1}^{m} \frac{1}{(1-x_\mu^2)[P_m^{(\alpha,\beta)\prime}(x_\mu)]^2}\\
&\quad= \frac{m!\,\Gamma(m+\alpha+\beta+1)}{\Gamma(m+\alpha+1)\,\Gamma(m+\beta+1)}\, 2^{-\alpha-\beta-1} \sum_{\mu=1}^{m} \lambda_\mu\\
&\quad= \frac{m!\,\Gamma(m+\alpha+\beta+1)}{\Gamma(m+\alpha+1)\,\Gamma(m+\beta+1)}\, \frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\\
&\quad= O(1) \qquad \text{as} \quad m\to\infty.
\end{aligned}$$

Thus it follows for $|x|<1$,

$$\|N_{mx}^{(\alpha,\beta)}\| = O(1)[P_m^{(\alpha,\beta)}(x)]^2 \qquad \text{as} \quad m\to\infty,$$

and we conclude the following (cf. Szegö [9, Theorem 7.32.1])

Lemma 2. *For the functionals $N_{mx}^{(\alpha,\beta)}$, one has the estimate*

$$\|N_{mx}^{(\alpha,\beta)}\| = o(1) \begin{cases} \textit{if} & |x|<1,\ \alpha,\beta>-1,\\ \textit{if} & |x|\leqslant 1,\ \alpha,\beta>-1 \ \textit{and}\ \max(\alpha,\beta)<0. \end{cases}$$

Altogether it follows

Theorem 2. *For the Hermite–Fejér functionals $F_{mx}^{(\alpha,\beta)}$, one has the estimates as $m\to\infty$*

$$\begin{aligned}
\|F_{mx}^{(\alpha,\beta)}\| &= 1 + O(1)\{P_m^{(\alpha,\beta)}(x)\}^2\\
&= 1 + o(1) \begin{cases} \textit{if} & |x|<1,\ \alpha,\beta>-1.\\ \textit{if} & |x|\leqslant 1,\ \alpha,\beta>-1 \ \textit{and}\ \max(\alpha,\beta)<0. \end{cases}
\end{aligned}$$

4. The Convergence of Hermite–Fejér Interpolation at Jacobi Nodes

According to a well-known result of Szegö [9, p. 339 f.] the Hermite–Fejér operators $F_m^{(\alpha,\beta)}$ are positive for all $m = 1, 2, \ldots,$ iff $\max(\alpha, \beta) \leqslant 0$. Therefore the Bohman–Korovkin theorem may be applied to the sequence $F_m^{(\alpha,\beta)}$ only for those values of α, β. But we may instead consider the positive functional $F_{mx}^{(\alpha,\beta)} - N_{mx}^{(\alpha,\beta)}$. Then we have for $|x| \leqslant 1$

$$|(N_{mx}^{(\alpha,\beta)} g_x)(x)| \leqslant \|N_{mx}^{(\alpha,\beta)}\| \, \|g_x\| \leqslant 4 \, \|N_{mx}^{(\alpha,\beta)}\|.$$

Now we apply the quantitative version of the Bohman–Korovkin theorem (de Vore [10, Theorem 2.3, p. 28 f.]) to $F_{mx}^{(\alpha,\beta)} - N_{mx}^{(\alpha,\beta)}$. Setting

$$\alpha_m^2(x) := (F_{mx}^{(\alpha,\beta)} g_x)(x) + 4 \, \|N_{mx}^{(\alpha,\beta)}\|,$$

$$e_0 : t \to 1,$$

we get since

$$F_m^{(\alpha,\beta)} e_0 = e_0$$

the following estimate

$$|(F_m^{(\alpha,\beta)} f)(x) - f(x)| \leqslant \{ |f(x)| + 1 \} \, \|N_{mx}^{(\alpha,\beta)}\| + \{ 1 + \|N_{mx}^{(\alpha,\beta)}\| + (1 + \|N_{mx}^{(\alpha,\beta)}\|)^{1/2} \} \, \omega(f, |\alpha_m(x)|).$$

We now apply Theorem 1, Lemma 2, and Theorem 2 getting

$$\alpha_m^2(x) = o(1) \begin{cases} \text{if} & |x| < 1, \alpha, \beta > -1, \\ \text{if} & |x| \leqslant 1, \alpha, \beta > -1 \text{ and } \max(\alpha, \beta) < 0 \end{cases}$$

and

$$|(F_m^{(\alpha,\beta)} f)(x) - f(x)| = o(1) \begin{cases} \text{if} & |x| < 1, \alpha, \beta > -1, \\ \text{if} & |x| \leqslant 1, \alpha, \beta > -1 \text{ and } \max(\alpha, \beta) < 0. \end{cases}$$

We point to the fact that the above estimations moreover show that the convergence is uniform in every subinterval $[-1+\delta, 1-\delta]$, $\delta > 0$, if $\alpha, \beta > -1$, and in the whole interval $[-1, 1]$ if $\max(\alpha, \beta) < 0$.

Theorem 3. *The sequence of Hermite–Fejér interpolation polynomials* $F_m^{(\alpha,\beta)} f$ *converges for every* $f \in C[-1, 1]$ *to* f

(i) *pointwise for all* x *with* $|x| < 1$, *if* $\alpha, \beta > -1$,

(ii) *uniform for all* x *with* $|x| \leqslant 1 - \delta$, $\delta > 0$, *if* $\alpha, \beta > -1$,

(iii) *uniform for all* x *with* $|x| \leqslant 1$, *if* $\alpha, \beta > -1$ *and* $\max(\alpha, \beta) < 0$.

5. Hermite–Fejér Interpolation at Jacobi Nodes of Multiplicity 4

The Hermite–Fejér interpolation problem has been generalized in the following way (cf. [2, 4, 5]):

For the zeros of the Jacobi polynomial $P_m^{(\alpha,\beta)}$, we consider the operator

$$K_m^{(\alpha,\beta)}: C[-1, 1] \to \pi_{4m-1}$$

with

$$(K_m^{(\alpha,\beta)} f)(x) := \sum_{\mu=1}^{m} f(x_\mu)\, u_\mu^{(\alpha,\beta)}(x) \{l_\mu(x)\}^4,$$

where l_μ as above and

$$\begin{aligned} u_\mu^{(\alpha,\beta)}(x) := {} & 1 - 2\frac{\gamma+\delta x_\mu}{1-x_\mu^2}(x-x_\mu) + \frac{11}{6}\left(\frac{\gamma+\delta x_\mu}{1-x_\mu^2}\right)^2 (x-x_\mu)^2 \\ & + \frac{1}{6}\left(\frac{x-x_\mu}{1-x_\mu^2}\right)^2 \left(1 - \frac{2\gamma+(2\delta-1)\,x_\mu}{1-x_\mu^2}(x-x_\mu)\right) \\ & \times [4(M-\delta)(1-x_\mu^2) - 8x_\mu(\gamma+\delta x_\mu)] \\ & - \left[\left(\frac{\gamma+\delta x_\mu}{1-x_\mu^2}\right)^3 + \frac{1}{3}x_\mu \frac{(\gamma+\delta x_\mu)^2}{(1-x_\mu^2)^3} \right. \\ & \left. + \frac{1}{6}(\delta+2)\frac{\gamma+\delta x_\mu}{(1-x_\mu^2)^2}\right](x-x_\mu)^3, \\ \gamma := {} & \alpha-\beta, \\ \delta := {} & \alpha+\beta+2, \\ M := {} & m(m+\alpha+\beta+1). \end{aligned}$$

We point to the fact that $K_m^{(\alpha,\beta)} f$ is nothing else but the "derivative-free" part of the generalized Hermite interpolation polynomial to f at the nodes x_μ with multiplicity 4. The convergence problem for the generalized Hermite–Fejér interpolation may be treated with methods which are similar to those used above.

First, we show convergence for the test polynomial g_x. We have

$$\begin{aligned} (K_m^{(\alpha,\beta)} g_x)(x) = {} & \{P_m^{(\alpha,\beta)}(x)\}^2 \sum_{\mu=1}^{m} \frac{1-x_\mu^2 - 2(\gamma+\delta x_\mu)(x-x_\mu)}{(1-x_\mu^2)[P_m^{(\alpha,\beta)\prime}(x_\mu)]^2} \{l_\mu(x)\}^2 \\ & + \{P_m^{(\alpha,\beta)}(x)\}^4 \sum_{\mu=1}^{m} \frac{1}{(1-x_\mu^2)^2\,[P_m^{(\alpha,\beta)\prime}(x_\mu)]^4} \left\{\frac{11}{6}(\gamma+\delta x_\mu)^2\right. \end{aligned}$$

$$+\frac{1}{6}\left(1-\frac{2\gamma+(2\delta-1)\,x_\mu}{1-x_\mu^2}\,(x-x_\mu)\right)$$
$$\times\,[4(M-\delta)(1-x_\mu^2)-8x_\mu(\gamma+\delta x_\mu)]$$
$$-\left(\frac{(\gamma+\delta x_\mu)^3}{1-x_\mu^2}+\frac{1}{3}\,x_\mu\,\frac{(\gamma+\delta x_\mu)^2}{1-x_\mu^2}\right.$$
$$\left.+\frac{1}{6}\,(\delta+2)(\gamma+\delta x_\mu)\right)(x-x_\mu)\Bigg\}.$$

As

$$\sum_{\mu=1}^{m}|l_\mu(x)|=O(\log m)$$

(cf. Szegö [9, p. 335 ff.]) we get for the first sum by a similar estimation as in Section 3

$$\{P_m^{(\alpha,\beta)}(x)\}^2\sum_{\mu=1}^{m}\frac{1-x_\mu^2-2(\gamma+\delta x_\mu)(x-x_\mu)}{(1-x_\mu^2)[P_m^{(\alpha,\beta)\prime}(x_\mu)]^2}\,\{l_\mu(x)\}^2$$
$$=\{P_m^{(\alpha,\beta)}(x)\}^2\,O(\log^2 m)$$
$$=\begin{cases} O\left(\dfrac{\log^2 m}{m}\right) & \text{as } m\to\infty \text{ if } |x|<1,\\[2ex] O(\log^2 m\cdot m^{2\max(\alpha,\beta,-0.5)}) & \text{as } m\to\infty \text{ if } |x|\leqslant 1.\end{cases}$$

For the second sum we note

$$\frac{1}{1-x_\mu^2}=O(m^2)$$

and on the other hand we compare to the Gauss–Jacobi weights (Szegö [9, (15.3.1)]) and get

$$\sum_{\mu=1}^{m}\frac{1}{(1-x_\mu^2)^2\,[P_m^{(\alpha,\beta)\prime}(x_\mu)]^4}=O(1)\sum_{\mu=1}^{m}\lambda_\mu^2.$$

From [9, (15.3.14)] we deduce

$$\sum_{\mu=1}^{m}\lambda_\mu^2=O(1)\sum_{\mu=1}^{m}\left(\frac{\mu^{4\alpha+2}}{m^{4\alpha+4}}+\frac{\mu^{4\beta+2}}{m^{4\beta+2}}\right)$$
$$=\begin{cases} O(m^{-1}) & \text{if } \min(\alpha,\beta)>-\dfrac{3}{4},\\[2ex] O(\log m\, m^{-1}) & \text{if } \min(\alpha,\beta)=-\dfrac{3}{4}\end{cases}$$

(cf. also [4]). Altogether we have as $m \to \infty$,

$$
\begin{aligned}
&(K_m^{(\alpha,\beta)} g_x)(x) \\
&\quad = \{P_m^{(\alpha,\beta)}(x)\}^2\, O(\log^2 m) + \{P_m^{(\alpha,\beta)}(x)\}^4\, O(m^2) \sum_{\mu=1}^{m} \lambda_\mu^2 \\
&\quad = \begin{cases} O\left(\dfrac{\log^2 m}{m}\right) + O\left(\dfrac{\log m}{m}\right) & \text{if } \min(\alpha,\beta) \geqslant -\dfrac{3}{4},\ |x|<1, \\ m^{2\max(\alpha,\beta,-0.5)}\, O(\log m) + m^{4\max(\alpha,\beta,-0.5)+1}\, O(\log m) & \\ & \text{if } \min(\alpha,\beta) \geqslant -\dfrac{3}{4},\ |x| \leqslant 1. \end{cases} \\
&\quad = o(1) \begin{cases} \text{if} & \min(\alpha,\beta) \geqslant -\frac{3}{4},\ |x|<1, \\ \text{if} & (\alpha,\beta) \in [-\frac{3}{4}, -\frac{1}{4}),\ |x| \leqslant 1. \end{cases}
\end{aligned}
$$

So we get the following convergence theorem:

THEOREM 4. *The Hermite–Fejér interpolation of multiplicity* 4 *applied to the test function* g_x *is convergent in the following sense*

$$
(K_m g_x)(x) = o(1) \begin{cases} \textit{if} & |x|<1,\ \alpha,\beta \geqslant -\frac{3}{4}, \\ \textit{if} & |x| \leqslant 1,\ (\alpha,\beta) \in [-\frac{3}{4}, -\frac{1}{4}). \end{cases}
$$

To show the asymptotic positivity of the functionals $K_{mx}^{(\alpha,\beta)}$ with

$$
K_{mx}^{(\alpha,\beta)} f := (K_m^{(\alpha,\beta)} f)(x)
$$

we analyse the position of the zeros of

$$
\begin{aligned}
u_x(t) = 1 &- 2\frac{\gamma+\delta t}{1-t^2}(x-t) + \frac{11}{6}\left(\frac{\gamma+\delta t}{1-t^2}\right)^2 (x-t)^2 \\
&+ \frac{1}{6}\left(\frac{x-t}{1-t^2}\right)^2 \left\{1 - \frac{2\gamma+(2\delta-1)t}{1-t^2}(x-t)\right\} \\
&\times [4(M-\delta)(1-t^2) - 8t(\gamma+\delta t)] \\
&- \left[\left(\frac{\gamma+\delta t}{1-t^2}\right)^3 + \frac{1}{3}t\frac{(\gamma+\delta t)^2}{(1-t^2)^3} + \frac{1}{6}(\delta+2)\frac{\gamma+\delta t}{(1-t^2)^2}\right](x-t)^3;
\end{aligned}
$$

here we have $\delta > 0$ and without loss of generality $\gamma \geqslant 0$. The demand

$$
u_x(t) = 0
$$

is equivalent to

$$\frac{1}{t-x}+\frac{1}{6}\frac{t-x}{(1-t^2)^2}\left\{1+\frac{2\gamma+(2\delta-1)\,t}{1-t^2}(t-x)\right\}$$
$$\times[4(M-\delta)(1-t^2)-8t(\gamma+\delta t)]$$
$$=-2\frac{\gamma+\delta t}{1-t^2}-\frac{11}{6}\left(\frac{\gamma+\delta t}{1-t^2}\right)(t-x)$$
$$-\left[\left(\frac{\gamma+\delta t}{1-t^2}\right)^3+\frac{1}{3}t\frac{(\gamma+\delta t)}{(1-t^2)^3}+\frac{1}{6}(\delta+2)\frac{\gamma+\delta t}{(1-t^2)^2}\right](t-x)^2. \tag{5.1}$$

Let x be fixed with $|x|<1$. Then $\varepsilon_1>0$ exists such that

$$1+\frac{2\gamma+(2\delta-1)\,t}{1-t^2}(t-x)\geqslant 0 \qquad \text{if} \quad |x-t|\leqslant\varepsilon_1.$$

As $M=O(m^2)$, there exists $\varepsilon_2>0$ such that

$$4(M-\delta)(1-t^2)-8t(\gamma+\delta t)\geqslant 0$$

if $|x-t|\leqslant\varepsilon_2$ and $m\geqslant m_0(\varepsilon_2)$. Therefore $\tilde\varepsilon=\tilde\varepsilon(x,\alpha,\beta)>0$ exists such that for $m\geqslant m_0(\tilde\varepsilon)$ the left side in (5.1) is negative if $x-\tilde\varepsilon\leqslant t<x$ and positive if $x<t\leqslant x+\tilde\varepsilon$ and unbounded if one approaches the point x. As the right side is bounded at x, there exists $\varepsilon=\varepsilon(x,\alpha,\beta)>0$ such that for $|x-t|\leqslant\varepsilon$ and $m\geqslant m_0(\varepsilon)$ no zero of u_x exists. Therefore from $u_\mu^{(\alpha,\beta)}(x)<0$ the inequality $|x-x_\mu|>\varepsilon$ follows. Altogether we have as $m\to\infty$,

$$\|K_{mx}^{(\alpha,\beta)}\|=1-2\sum_{\substack{\mu=1\\u_\mu(x)<0}}^{m}u_\mu(x)\{l_\mu(x)\}^4$$

and in view of $(1-x_\mu^2)^{-1}=O(m^2)$, we get

$$\|K_{mx}^{(\alpha,\beta)}\|=1+\{P_m^{(\alpha,\beta)}(x)\}^4\,O(m^2)\sum_{\mu=1}^{m}\frac{1}{(1-x_\mu^2)^2\,[P_m^{(\alpha,\beta)\prime}(x_\mu)]^4}$$
$$=1+\{P_m^{(\alpha,\beta)}(x)\}^4\,O(m^2)\sum_{\mu=1}^{m}\lambda_\mu^2.$$

Now we use our estimations from the proof of Theorem 4 getting

$$\|K_{mx}^{(\alpha,\beta)}\|=1+o(1)\begin{cases}\text{pointwise} & \text{if } |x|<1 \text{ and } \min(\alpha,\beta)\geqslant-\frac{3}{4},\\ \text{uniform} & \text{if } |x|\leqslant 1-\delta,\ \delta>0 \text{ and } \min(\alpha,\beta)\geqslant-\frac{3}{4},\\ \text{uniform} & \text{if } |x|\leqslant 1 \text{ and } (\alpha,\beta)\in[-\frac{3}{4},-\frac{1}{4})^2.\end{cases}$$

THEOREM 5. *The functionals* $K_{mx}^{(\alpha,\beta)}$ *are asymptotically positive as* $m \to \infty$,

(i) *for* $|x| < 1$ *if* $\min(\alpha, \beta) \geqslant -\frac{3}{4}$,

(ii) *for* $|x| \leqslant 1$ *if* $(\alpha, \beta) \in [-\frac{3}{4}, -\frac{1}{4})^2$.

From Theorems 4 and 5 we now conclude the convergence theorem for the Hermite–Fejér interpolation with Jacobi nodes of multiplicity 4 additionally noting that the convergence is uniform in compact subintervals.

THEOREM 6. *The sequence of Hermite–Fejér interpolation polynomials with Jacobi nodes of multiplicity* 4 *converges for every* $f \in C[-1, 1]$ *to* f

(i) *pointwise for all* x *with* $|x| < 1$, *if* $\min(\alpha, \beta) \geqslant -\frac{3}{4}$,

(ii) *uniform for all* x *with* $|x| \leqslant 1 - \delta$, $\delta > 0$, *if* $\min(\alpha, \beta) \geqslant -\frac{3}{4}$,

(iii) *uniform for all* x *with* $|x| \leqslant 1$, *if* $(\alpha, \beta) \in [-\frac{3}{4}, -\frac{1}{4})^2$.

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